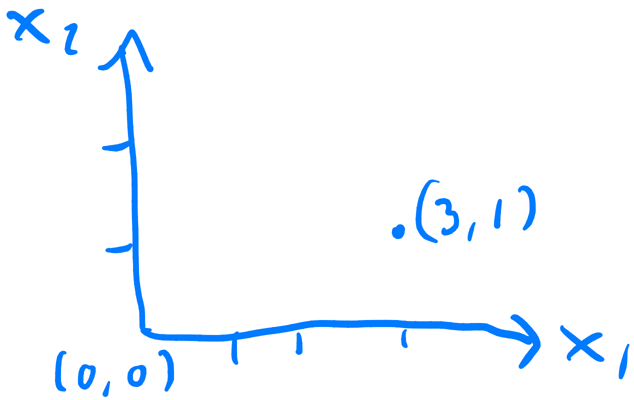
$$\boxed{\text{Week 1}}$$

I. Scalar and Vector Fields
Differential operators

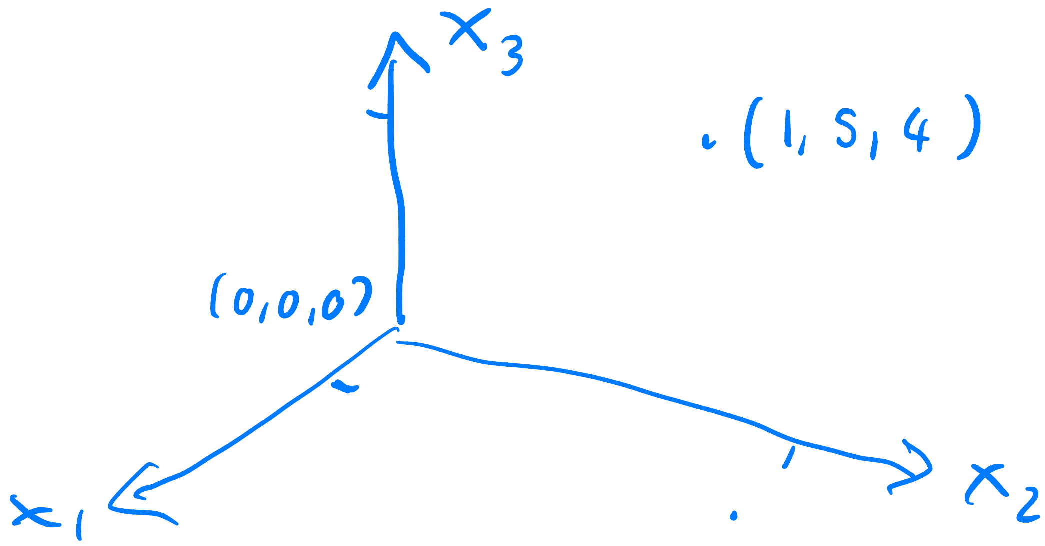
I.1 Review Euclidean Space

- We work in n-dimensional space $x = (x_1, x_2, \dots, x_n)$
- 2D: $v = (x_1, x_2) = (x, y)$
- 3D: $v = (x_1, x_2, x_3) = (x, y, z)$

$n = 2$



$n = 3$



- Scalar / dot product

$$x = (x_1, \ldots, x_n) \quad y = (y_1, \ldots, y_n)$$

$$x \cdot y = \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$$

- Cross product (3D)

$$x = (x_1, x_2, x_3)$$

$$y = (y_1, y_2, y_3)$$

$$x \times y = \left(x_2 y_3 - x_3 y_2, \; x_3 y_1 - x_1 y_3, \; x_1 y_2 - x_2 y_1 \right)$$

I.2. Review Scalar fields

A scalar field

$$F : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto F(x)$$

assigns a scalar value to every point

Examples: temperature, chemical concentration, pressure, probability distributions

Example

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad x \mapsto \sin(x_1^2 + x_2^2)$$

$$g : \mathbb{R}^3 \to \mathbb{R}, \quad x \mapsto \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Euclidean norm

Given a scalar field $f : \mathbb{R}^n \to \mathbb{R}$,
the **level set** of the value $c \in \mathbb{R}$ is

$$\{ x \in \mathbb{R}^n : f(x) = c \}$$

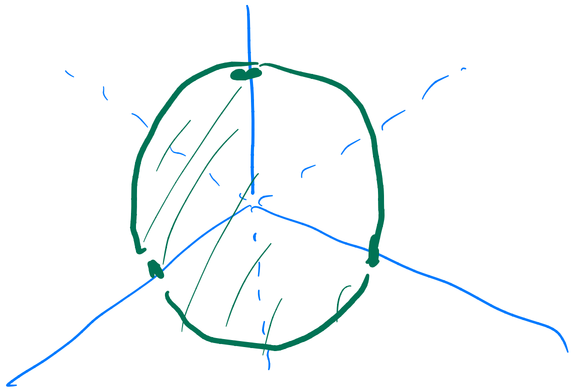
set notation

Example Consider the scalar field

$$h : \mathbb{R}^3 \to \mathbb{R}, \quad x \mapsto x_1^2 + x_2^2 + x_3^2$$

Level set of $c = 1$:

$$\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \}$$



unit sphere around origin

I.3 Review Vector Fields

A vector field assigns a vector to each point in space

$$F : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto (F_1(x), \ldots, F_n(x))$$

Examples: velocity, accelleration, Force fields, fluid flow

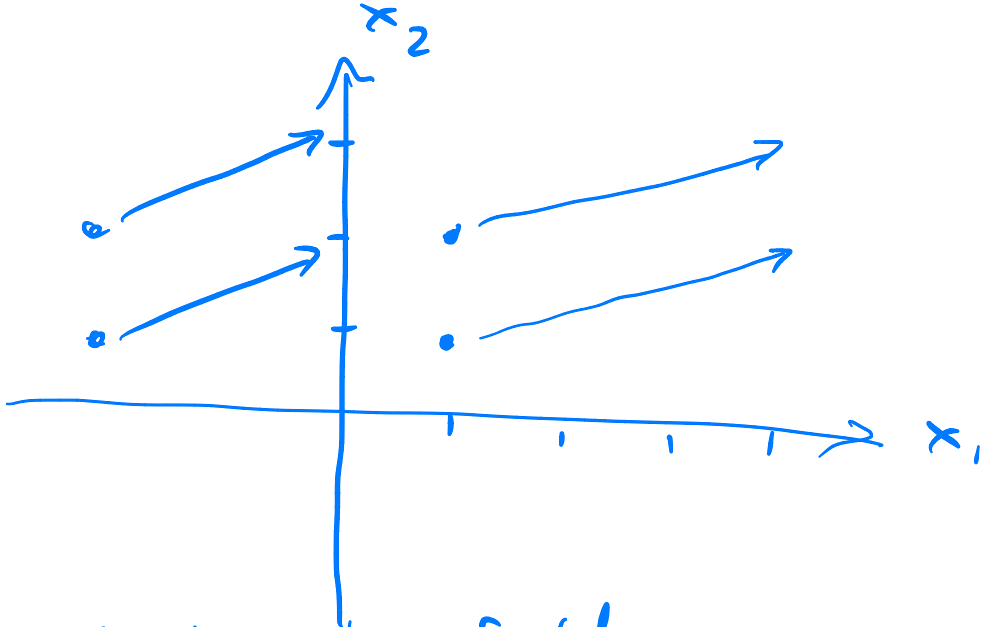
Examples:

$$F: \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \longmapsto (-x_2, x_1)$$

$$G: \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (2x_1, x_2 + x_3, x_1 x_2 x_3)$$

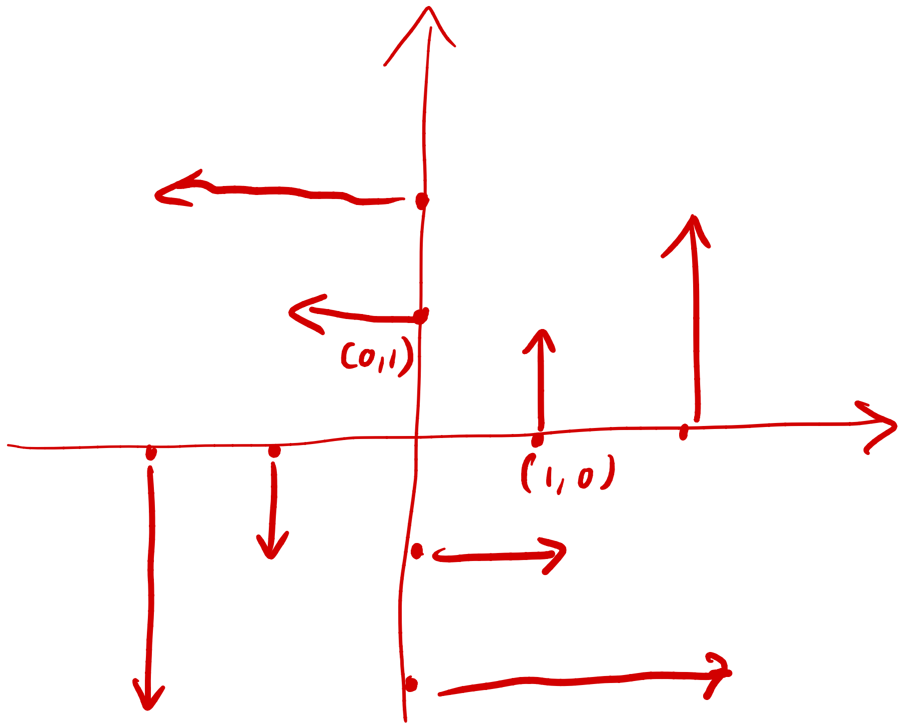
How to visulize vector fields?

Example $A(x_1, x_2) = (3, 1)$



Constant vector field
(constant flow)

Example $B(x_1, x_2) = (-x_2, x_1)$

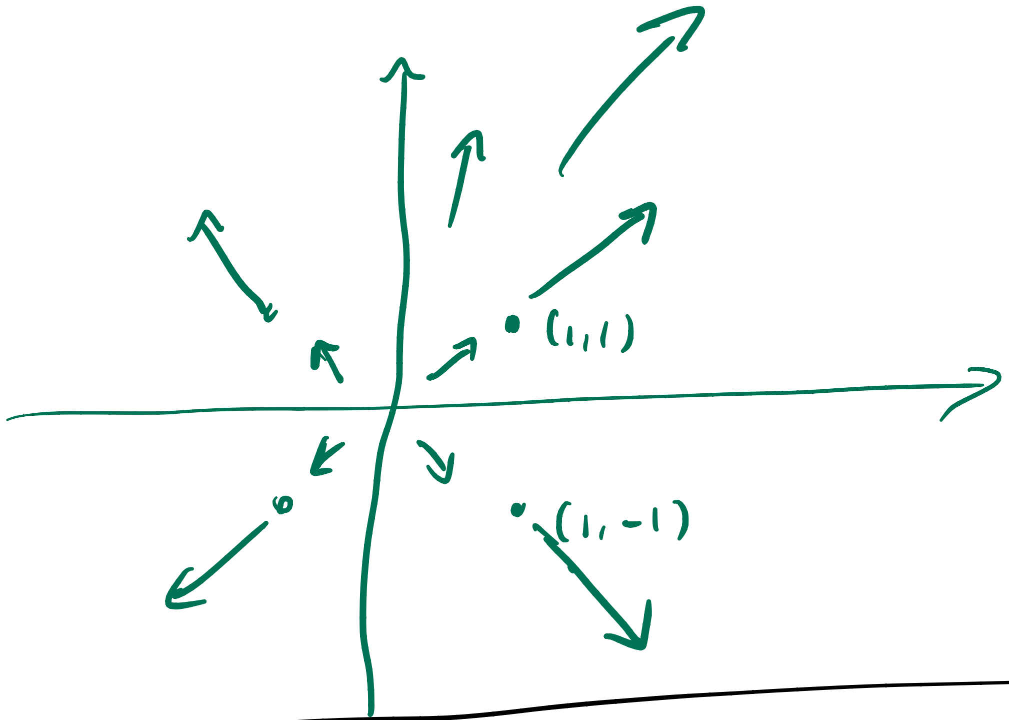


rotational vector field

(circular flow)

counterclockwise orientation
magnitude increases away
from the origin

Example: $C(x_1, x_2) = (x_1, x_2)$



Source vector field

(describes a flow away from the origin)

We have seen scalar fields $F: \mathbb{R}^n \to \mathbb{R}$

vector fields $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$

Similarly matrix fields $\underline{\underline{F}}: \mathbb{R}^n \to \mathbb{R}^{n \times n}$

We have discussed fields over $\mathbb{R}^n$. More generally over open sets $\Omega \subseteq \mathbb{R}^n$

I.4 Review: Gradient, Hessian, Laplacian

If $f: \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the gradient is the vector field

$$\operatorname{grad} f = \left(\partial_{x_1} f, \ \partial_{x_2} f, \ \ldots, \ \partial_{x_n} f \right)$$

Notation: Using the nabla symbol ∇

$$\nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix} \qquad \operatorname{grad} f = \nabla f = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_n} f \end{pmatrix}$$

Here, for convenience, we neglect the difference between column and row vectors

Example

$$f(x_1, x_2) = x_1^3 + x_1 x_2^5$$

$$\text{grad } f(x_1, x_2) = \left(3x_1^2 + x_2^5 ,\ x_1 \cdot 5x_2^4 \right)$$

$$g(x_1, x_2) = x_1 x_2$$

$$\text{grad } g(x_1, x_2) = (x_2, x_1)$$

- Geometrically, grad f points in the direction of the steepest increase of f

- The <u>directional derivative</u> of f in direction $v \in \mathbb{R}^n$ is:

$$D_v f = \operatorname{grad} f \cdot v = \partial_{x_1} f \cdot v_1 + \dots + \partial_{x_n} f \cdot v_n$$

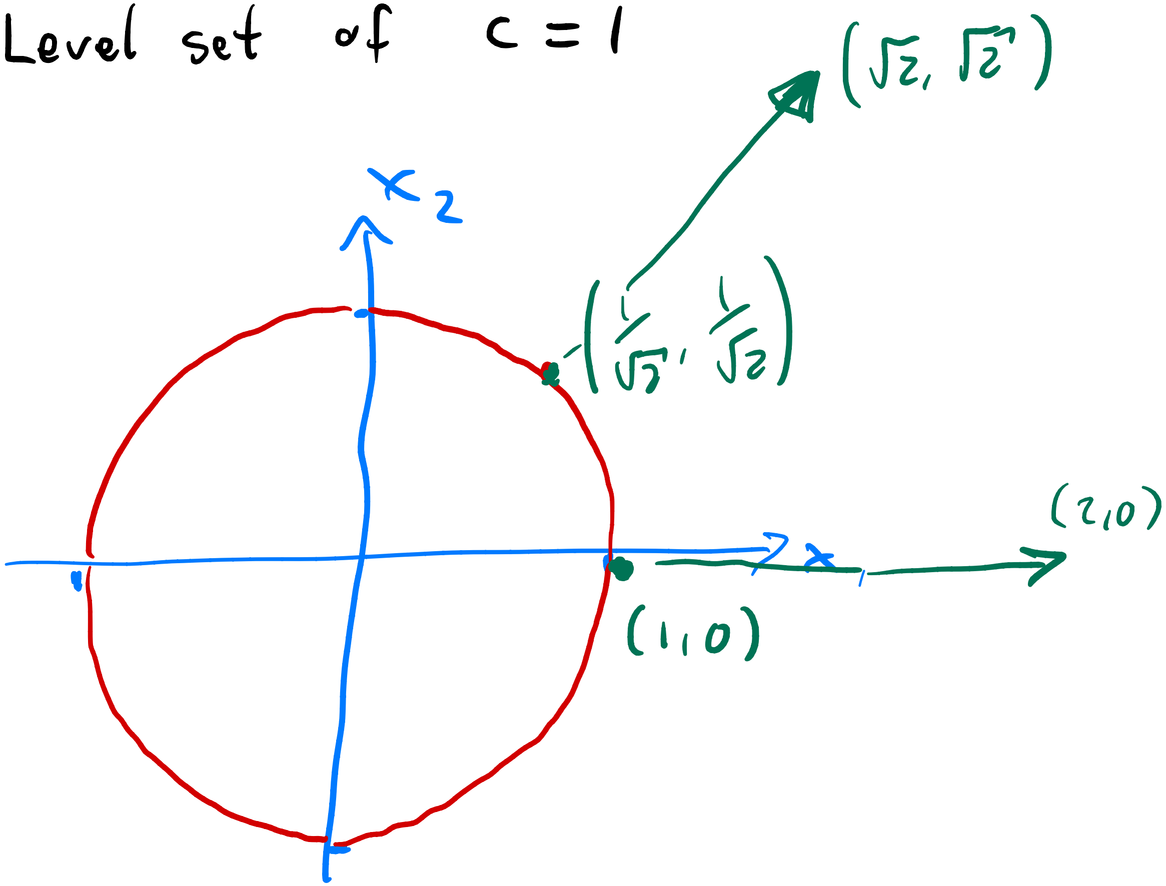
The value $D_v f$ tells us how f changes as we move in direction v

- Along any level set line, the gradient is orthogonal

Example $f(x_1, x_2) = x_1^2 + x_2^2$

$$\text{grad} f(x_1, x_2) = (2x_1, 2x_2)$$

Level set of $c = 1$



If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field,
then the Hessian of f is a matrix field

$$\nabla^2 f = \begin{pmatrix} \partial_1 \partial_1 f & \partial_1 \partial_2 f & \cdots & \partial_1 \partial_n f \\ \partial_2 \partial_1 f & \partial_2 \partial_2 f & \cdots & \\ \vdots & & & \\ \partial_n \partial_1 f & \cdots & \cdots & \partial_n \partial_n f \end{pmatrix}$$

- The Hessian is symmetric if all 2nd derivatives are continuous
- Physical/geometric interpretation contains information on the curvature of the scalar field

If $f: \mathbb{R}^n \to \mathbb{R}$ is a scalar field, then the Laplacian of f is the scalar field:

$$\Delta f = \sum_{i=1}^{n} \partial_i \partial_i f = \partial_1 \partial_1 f + \partial_2 \partial_2 f + \cdots + \partial_n \partial_n f$$

- Sum of the diagonal entries of the Hessian
- Physically relevant in modeling diffussion

Poisson problem: $-\Delta u = f$ (differential equation)

unknown scalar field

known scalar field

Example: $f(x,y,z) = xy^3 e^z$

$$\operatorname{grad} f(x) = \left(y^3 e^z ,\ 3xy^2 e^z ,\ xy^3 e^z \right)$$

$$\partial_x \partial_x f = 0, \quad \partial_x \partial_y f = 3y^2 e^z, \quad \partial_x \partial_z f = y^3 e^z$$

$$\partial_y \partial_y f = 6xye^z, \quad \partial_y \partial_z f = 3xy^2 e^z, \quad \partial_z \partial_z f = xy^3 e^z$$

$$\nabla^2 f = \begin{pmatrix} 0 & 3y^2e^z & y^3e^z \\ 3y^2e^z & 6xye^z & 3xy^2e^z \\ y^3e^z & 3xy^2e^z & xy^3e^z \end{pmatrix}$$

$$\Delta f = 0 + 6xye^z + xy^3e^z$$
$$= e^z xy\left(6 + y^2\right)$$

Example: $f(x) = \|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ $\quad x \in \mathbb{R}^n$

First derivatives

$$\partial_i f(x) = \partial_i \left(\sum_{i=1}^n x_i^2\right)^{1/2} = \frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)^{-1/2} \cdot 2x_i$$

$$= \frac{1}{\|x\|} x_i = \frac{x_i}{\|x\|}$$

chain rule

derivatives of composed functions

Notice the singularity at $x = 0$

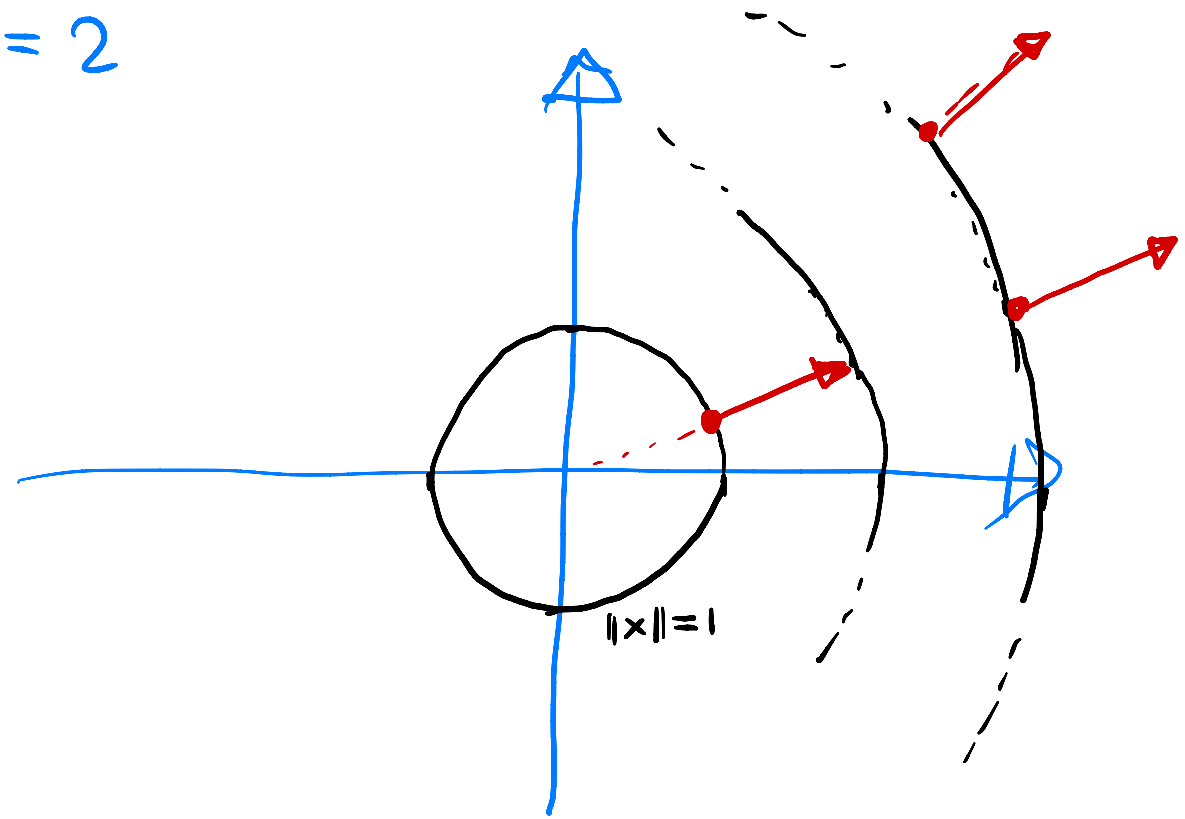
Not removable

$$\operatorname{grad} f(x) = \left(\frac{x_1}{\|x\|}, \frac{x_2}{\|x\|}, \dots, \frac{x_n}{\|x\|}\right) = \frac{x}{\|x\|}$$

unit vector

length $= 1$

$n = 2$



NB:

- gradient is orthogonal to the level sets
- it points in the direction of steepest increase

Second partial derivatives of $f(x) = \|x\|$. Case distinction

$$\partial_i \partial_i f(x) = \partial_i \left(\frac{x_i}{\|x\|} \right) = \partial_i \left(x_i \cdot \|x\|^{-1} \right) = \frac{1}{\|x\|} + x_i \cdot \frac{-1}{\|x\|^2} \left(\partial_i \|x\| \right)$$

$$= \frac{1}{\|x\|} + x_i \cdot \frac{-1}{\|x\|^2} \cdot \frac{x_i}{\|x\|} = \frac{1}{\|x\|} - x_i^2 \cdot \frac{1}{\|x\|^3}$$

$$\partial_j \partial_i f(x) = \partial_j\left(\frac{x_i}{\|x\|}\right) = 0 + x_i \cdot \frac{-1}{\|x\|^2}\left(\partial_j \|x\|\right)$$

$j \neq i$

$$= x_i \cdot \frac{-1}{\|x\|^2} \frac{x_j}{\|x\|} = -\frac{x_i x_j}{\|x\|^3}$$

All partial derivatives of order 2, we can build the <u>Hessian matrix</u>

We want the Laplacian too:

$$\Delta f(x) = \sum_{i=1}^{n} \partial_i \partial_i f(x) = \sum_{i=1}^{n} \frac{1}{\|x\|} - \frac{x_i^2}{\|x\|^3} = \frac{n}{\|x\|} - \frac{\sum_{i=1}^{n} x_i^2}{\|x\|^3}$$

$$= \frac{n}{\|x\|} - \frac{\|x\|^2}{\|x\|^3} = \frac{n-1}{\|x\|}$$

I.5 Divergence

Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (F_1(x), \dots, F_n(x))$ the **divergence** is the vector field

$$\operatorname{div} F = \sum_{i=1}^{n} \partial_i F_i = \partial_1 F_1 + \partial_2 F_2 + \dots + \partial_n F_n$$

- Formally, $\operatorname{div} F = \nabla \bullet F$
- The Laplacian is the divergence of the gradient: $\Delta f = \operatorname{div} \operatorname{grad} f$

Example

$$\operatorname{div}\left(x_1^2 x_2,\ x_2^3,\ e^{x_3}\right) = 2x_1x_2 + 3x_2^2 + e^{x_3}$$

I. 6 Rotation or curl of vector fields

If $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a 3D vector field, $F = (F_1, F_2, F_3)$, then the curl / rotation is a 3D vector field

$$\operatorname{curl} F = \operatorname{rot} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

Formally, $\operatorname{curl} F = \nabla \times F$

Only works in 3D. There is a rotation in 2D:

If $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a 2D vector field, then the rotation / curl of F is a scalar field

$$\operatorname{curl} F = \operatorname{rot} F = \partial_1 F_2 - \partial_2 F_1$$

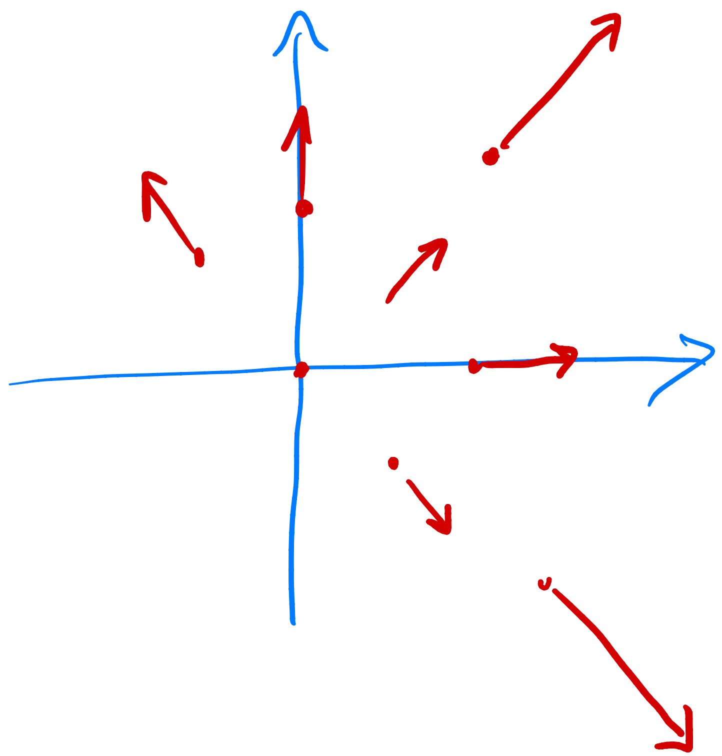
Motivation: We formally extend the vector field with a third coordinate $F_3 = 0$.

$$\tilde{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \Rightarrow \operatorname{curl} F = \begin{pmatrix} \cancel{\partial_2 F_3} - \cancel{\partial_3 F_2} \\ \cancel{\partial_3 F_1} - \cancel{\partial_1 F_3} \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\partial_2 F_1 + \partial_1 F_2 \end{pmatrix}$$

The 2D curl is a restriction of the 3D curl

<u>Examples</u> : Divergence measures the presence of sinks and sources
Rotation measures the presence of a spin

$$A(x_1, x_2) = (x_1, x_2)$$

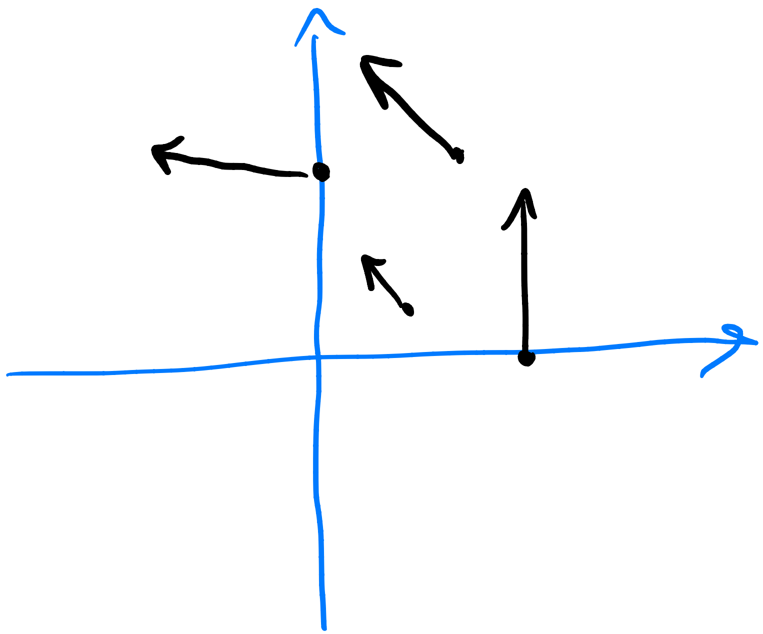


$$\operatorname{div} A(x_1, x_2) = \partial_1 x_1 + \partial_2 x_2 = 2$$

$$\operatorname{rot} A(x_1, x_2) = \partial_1 x_2 - \partial_2 x_1 = 0$$

rotation-free

$$B(x_1, x_2) = (-x_2, x_1)$$



$$\operatorname{div} B(x_1, x_2) = \partial_1 x_2 - \partial_2 x_1 = 0$$

divergence - free

$$\operatorname{rot} B(x_1, x_2) = \partial_1 x_1 - \partial_2(-x_2)$$

$$= 2$$